

IR Divergence and Anomalous Temperature Dependence of the Condensate in the Quenched Schwinger Model

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Abstract

The Schwinger model is used to study the artifacts of quenching in a controlled way. The model is solved on a finite-temperature cylinder of circumference $\beta = 1/T$ with bag-inspired local boundary conditions at the two ends $x^1 = 0$ and $x^1 = L$ which break the γ_5 -invariance and thus play the role of a small quark mass. The quenched chiral condensate is found to diverge exponentially as $L \rightarrow \infty$, and to diverge (rather than melt as for $N_f \geq 1$) if the high-temperature limit $\beta \rightarrow 0$ is taken at finite box-length L . We comment on the generalization of our results to the massive quenched theory, arguing that the condensate is finite as $L \rightarrow \infty$ and proportional to $1/m$ up to logarithms.

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I. INTRODUCTION

Simulations of lattice QCD often use the quenched approximation in which the quark determinant is replaced by a constant [1]. While it is clear that the quenched theory is unphysical [2], it is less certain what pathologies are introduced in Euclidean correlation functions. Considerations based on quenched chiral perturbation theory suggest that the chiral limit is singular [3]. For example, the quark condensate is predicted to diverge in this limit: $\langle \bar{\psi}\psi \rangle \propto m^{-\delta}$, with $\delta \approx 0.1$ ¹. There is some numerical evidence supporting the predictions of quenched chiral perturbation theory, but it is not yet definitive [5].

Given this situation, it is interesting to explore the effect of quenching in a completely controlled environment where calculations can be done analytically. In this paper we undertake such an investigation for the Schwinger model (QED in two dimensions with N_f massless fermions [6]). This has been frequently used as a toy model for QCD(4), since it shares the properties of confinement and dynamical mass generation [7]. In particular, we have calculated the chiral condensate in the massless quenched theory with chiral symmetry breaking introduced by the spatial boundary conditions. The chirality violating parameter is $1/L$ (L being the box length) instead of the quark mass m . We also introduce thermal boundary conditions in the Euclidean time direction. In this way we can formally work with the massless theory throughout (the finite spatial and temporal extent of the manifold act as an infrared regulator), and maintain full analytic control. In particular, the result for the theory with positive N_f quantized with these boundary conditions can be analytically continued to the quenched limit, $N_f = 0$ ². We are able to calculate the condensate in the quenched theory at any spatial location as a function of L and the inverse temperature β .

A more straightforward approach would be to quantize the massive theory with an arbitrary number of flavors in a finite box, and then take the limits $N_f \rightarrow 0$, $L \rightarrow \infty$ and $m \rightarrow 0$ sequentially (in this particular order). The difficulty here is that even in the case of the massive QED(2) only an approximate solution exists [8]. Since analytical continuation in N_f is, in general, arbitrarily sensitive to small variations in input data, one is on more secure ground using an exact solution. The drawback in our approach of breaking the chiral symmetry by boundary conditions is that the last two limits mentioned above ($L \rightarrow \infty$ and $m \rightarrow 0$) are effectively being taken simultaneously.

There have been several previous investigations of the quenched Schwinger model, both analytical and numerical. Casher and Neuberger argue that the spectral density of the Dirac operator at zero eigenvalue is non-zero [9] — an observation which, if we were in four dimensions, would indicate spontaneous chiral symmetry breaking. Introducing an ad-hoc infrared regulator mass μ_{IR} , Carson and Kenway [10] and Grandou [11] conclude, using bosonization methods, that the quenched condensate actually diverges exponentially as the regulator is removed:

¹This divergence is not related to that due to exact zero modes in topologically non-trivial backgrounds, an effect which is suppressed in the infinite volume limit [4].

²In the presence of an infrared regulator, this is equivalent to directly calculating correlation functions in the quenched approximation, i.e. without the determinant.

$$\langle \psi^\dagger \psi \rangle \sim \mu_{\text{IR}} \exp \left(\frac{e^2}{2\pi\mu_{\text{IR}}^2} \right) . \quad (1)$$

Smilga reaches a similar conclusion based on an analysis of the eigenvalue spectrum of the Dirac operator [12]. Our calculation shows that the result (1) is just a specific form of the generic combination “power-law fall-off times exponential divergence”, different versions of which we find as exact asymptotic results when the boundaries are sent to infinity in various ways. Our approach uses a finite box with specific chirality breaking boundary conditions rather than an ad-hoc IR-regulator and allows for a generalization to finite temperatures.

The following section summarizes previous work on the Schwinger model with chirality breaking boundary conditions [13–15]. We then describe the analytic continuation to $N_f = 0$, and study its behavior in the interesting limits. In sec. IV we elucidate the origin of the IR divergences that we find. Section V is devoted to a more speculative discussion of the IR divergences in the massive quenched theory. We conclude with a summary of our results.

II. CHIRAL CONDENSATE WITH ARBITRARY NUMBER OF FLAVORS

A. Manifold Parameters and Abbreviations

The Euclidean Schwinger model (massless QED in $d = 2$ dimensions)

$$S[A, \psi^\dagger, \psi] = S_B[A] + S_F[A, \psi^\dagger, \psi] \\ S_B = \frac{1}{4} \int_M F_{\mu\nu} F_{\mu\nu} \quad , \quad S_F = \sum_{n=1}^{N_f} \int_M \psi_n^\dagger \not{D} \psi_n \quad (2)$$

is studied on the manifold

$$M = [0, \beta] \times [0, L] \quad \ni \quad (x^0, x^1) \quad (3)$$

with volume $V = \beta L$. In Euclidean time direction, the fields A and ψ are periodic and antiperiodic respectively with period β . Hence $x^0 = 0$ and $x^0 = \beta$ are identified (up to a sign) and the manifold is a cylinder. At the two spatial ends of the cylinder (i.e. at $x^1 = 0$ and $x^1 = L$) some specific chirality-breaking (XB-) boundary-conditions (which will be discussed below) are imposed. We use the one-flavor Schwinger mass

$$\mu \equiv \frac{|e|}{\sqrt{\pi}} \quad (4)$$

to simplify our notation, and introduce the dimensionless inverse temperature and box-length

$$\sigma \equiv \mu\beta \quad \lambda \equiv \mu L \quad (5)$$

as well as the dimensionless volume, aspect ratio, and spatial position

$$v = \sigma\lambda \quad , \quad \tau = \frac{\sigma}{2\lambda} \quad , \quad \xi = \frac{x^1}{L} . \quad (6)$$

B. Chirality Breaking Boundary Conditions

The proposal to study both QCD and the Schwinger model with chirality-breaking boundary-conditions goes back to Ref. [16]. The XB-boundary conditions can be motivated by requiring the operator $i\mathcal{D}$ to be symmetric under the scalar product $(\chi, \psi) := \int \chi^\dagger \psi d^2x$, which leads to the condition that the surface integral $\oint \chi^\dagger \not{n} \psi ds$ vanishes, where $\not{n} = n_\mu \gamma_\mu$ with n_μ denoting the outward oriented normal on the boundary. Imposing local linear boundary conditions which ensure this requirement amounts to having $\chi^\dagger \not{n} \psi = 0$ on the boundary for each pair. A sufficient condition to guarantee this is to require all modes to obey $\psi = B\psi$ on the boundary, where the boundary operator B (which is understood to act as the identity in flavor space) has to satisfy $B^\dagger \not{n} B = -\not{n}$ and $B^2 = 1$. In [13–15] the one-parameter family of solutions

$$B \equiv B_\theta := i\gamma_5 e^{\theta\gamma_5} \not{n} \quad (7)$$

was chosen, supplemented by suitable boundary conditions for the gauge-field. The γ_5 invariance of the theory is broken for all θ , thus making the N_f -flavor theory invariant under $SU(N_f)_V$ instead of $SU(N_f)_L \times SU(N_f)_R$. In physical terms, these boundary conditions prevent the $U(1)$ -current from leaking through the boundary, since $j \cdot n = \psi^\dagger \not{n} \psi = 0$ on ∂M . There are considerable differences regarding the spectrum of the Dirac operator in the theory with XB-boundary-conditions [13,14] as compared to the theory on the torus [17]:

- The Dirac operator has a discrete real spectrum which is *asymmetric* with respect to zero.
- The spectrum is empty at zero, i.e. the Dirac operator has *no zero modes*.
- The instanton number $q = e/(4\pi) \cdot \int \epsilon_{\mu\nu} F_{\mu\nu} = e/(2\pi) \cdot \int E \in \mathbf{R}$ is *not quantized*.

The first property already indicates that we are away from the usual Atiyah-Patodi-Singer index-theorem situation. The fact that the second property is fulfilled implies that the generating functional for the fermions in a given gauge-field background

$$Z_F[A, \eta^\dagger, \eta] = \frac{1}{N} \int \prod_{i=1}^{N_f} D\psi_{(i)}^\dagger D\psi_{(i)} e^{-\sum \int \psi_{(i)}^\dagger \mathcal{D} \psi_{(i)} + \sum \int \psi_{(i)}^\dagger \eta_{(i)} + \sum \int \eta_{(i)}^\dagger \psi_{(i)}} \quad (8)$$

does indeed simplify to the textbook formula

$$Z_F[A, \eta^\dagger, \eta] = \left(\frac{\det_\theta(\mathcal{D})}{\det_\theta(\not{\partial})} \right)^{N_f} e^{\int \eta^\dagger (\mathcal{D})^{-1} \eta} \quad (9)$$

from which the one-flavor condensate in a given background [no sum over (i)]

$$\langle \psi_{(i)}^\dagger(x) P_\pm \psi_{(i)}(x) \rangle_A = \frac{1}{Z_F} \frac{\delta^2}{\delta \eta_{(i)\pm}(x) \delta \eta_{(i)\pm}^\dagger(x)} Z_F \Big|_{\eta_\pm = \eta_\pm^\dagger = 0} \quad (10)$$

is computed. Here and in the following $P_\pm = 1/2 \cdot (1 \pm \gamma_5)$ denotes the projector on the two chiralities, where $\gamma_5 = \text{diag}(1, -1)$ in the chiral representation of the Dirac Clifford algebra.

C. General Result at Arbitrary Points

The condensate in the N_f -flavor Schwinger Model on a finite-temperature cylinder with the XB-boundary-conditions (7) at the two spatial ends was found to read [14,15]

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} = \pm \frac{e^{\pm \theta \cdot \cosh(\lambda \sqrt{N_f}(1-2\xi)/2) / \cosh(\lambda \sqrt{N_f}/2)}}{4\lambda} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \frac{\sin(\pi \xi) \cosh(\pi n \tau)}{\sin^2(\pi \xi) + \sinh^2(\pi n \tau)} \cdot \frac{\int_{-1/2}^{1/2} dc \cos(2\pi n c) \theta_3^{N_f}(c, i\tau)}{\int_{-1/2}^{1/2} dc \theta_3^{N_f}(c, i\tau)} \cdot \exp \left\{ \frac{1}{N_f} \sum_{n \geq 1} \left(1 - \cos(2\pi n \xi) \right) \left(\frac{\coth(\pi n \tau)}{n} - (n \rightarrow \sqrt{n^2 + N_f(\lambda/\pi)^2}) \right) \right\} \quad (11)$$

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} = \pm \frac{e^{\pm \theta \cdot \cosh(\lambda \sqrt{N_f}(1-2\xi)/2) / \cosh(\lambda \sqrt{N_f}/2)}}{2\sigma} \cdot \sum_{m \in \mathbb{Z}} (-1)^m \frac{1}{\sinh(\pi(m+\xi)/\tau)} \cdot \frac{\int_{-1/2}^{1/2} dc \cosh(2\pi(m+\xi)c/\tau) e^{-N_f \pi c^2/\tau} \theta_3^{N_f}(ic/\tau, i/\tau)}{\int_{-1/2}^{1/2} dc e^{-N_f \pi c^2/\tau} \theta_3^{N_f}(ic/\tau, i/\tau)} \cdot \exp \left\{ \frac{\pi}{N_f} \left(\frac{\xi(1-\xi)}{\tau} + \frac{\cosh(\lambda \sqrt{N_f}(1-2\xi)) - \cosh(\lambda \sqrt{N_f})}{\sigma \sqrt{N_f} \sinh(\lambda \sqrt{N_f})} \right) \right\} \cdot \exp \left\{ \frac{1}{N_f} \sum_{m \geq 1} \frac{\cosh(\pi m/\tau) - \cosh(\pi m(1-2\xi)/\tau)}{m \sinh(\pi m/\tau)} - (m \rightarrow \sqrt{m^2 + N_f(\sigma/2\pi)^2}) \right\}, \quad (12)$$

where the integration variable c represents the harmonic piece in the Hodge decomposition of the gauge potential, and the elliptic function θ_3 is defined as

$$\theta_3(u, \omega) = \sum_{n \in \mathbb{Z}} e^{2\pi i n u} q^{n^2} = 1 + 2 \sum_{n \geq 1} \cos(2\pi n u) q^{n^2} \quad (q \equiv e^{i\pi\omega}) \quad (13)$$

for the parameters $\omega = i\tau$ and $\omega = i/\tau$ (giving real nome $q \in]0, 1[$). The two forms (11, 12) are identical for any finite σ and λ , but they enjoy excellent convergence properties in the regimes $\tau \gg 1$ and $\tau \ll 1$, respectively ($\tau = \sigma/2\lambda$).

III. QUENCHED CHIRAL CONDENSATE

A. Quenched Chiral Condensate at Arbitrary Points

This paper is based on the observation that – even though certain expressions within formulas (11, 12) diverge as $N_f \rightarrow 0$ (for fixed σ, λ) – the product of all factors stays finite:

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} = \pm \frac{e^{\pm \theta}}{4\lambda \sin(\pi \xi)} \cdot \exp \left\{ \frac{\lambda^2}{2\pi} \sum_{n \geq 1} \left(1 - \cos(2\pi n \xi) \right) \left(\frac{\coth(\pi n \tau)}{\pi n^3} + \frac{\tau}{n^2 \sinh^2(\pi n \tau)} \right) \right\} \quad (14)$$

$$\begin{aligned}
\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} &= \pm \frac{e^{\pm\theta}}{2\sigma} \frac{\tau}{\sin(\pi\xi)} \cdot \exp \left\{ \frac{\pi\sigma^2}{12\tau^3} \xi^2 (1-\xi)^2 \right\} \cdot \\
&\exp \left\{ \frac{\sigma^2}{8\pi\tau} \sum_{m \geq 1} \frac{1}{m^2 \sinh^2(\pi m/\tau)} \left(1 - \xi \cosh\left(\frac{2\pi m(1-\xi)}{\tau}\right) - (1-\xi) \cosh\left(\frac{2\pi m\xi}{\tau}\right) \right) \right\} \cdot \\
&\exp \left\{ \frac{\sigma^2}{8\pi^2} \sum_{m \geq 1} \frac{1}{m^3 \sinh(\pi m/\tau)} \left(\cosh\left(\frac{\pi m}{\tau}\right) - \cosh\left(\frac{\pi m(1-2\xi)}{\tau}\right) \right) \right\}, \quad (15)
\end{aligned}$$

where the low- and high-temperature forms (14, 15) are identical for any finite σ, λ .

B. Specialization to the Quenched Case at Midpoints

To study chiral symmetry breaking with XB-boundary conditions, one has to determine whether the condensate in the bulk of the cylinder survives when the boundaries are sent to infinity. We choose to consider the midpoint ($\xi = 1/2$), and also to set the boundary parameter θ to zero, for which the resulting expressions are most simple. Other values of ξ and θ lead to the same conclusions. Specializing eqs. (14, 15) to $\xi = 1/2$ and $\theta = 0$, and rewriting them in terms of the variables $v = \sigma\lambda$ and $\tau = \sigma/2\lambda$, we find

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} = \pm \frac{\sqrt{\tau}}{2^{3/2}\sqrt{v}} \cdot \exp \left\{ \frac{v}{2\pi\tau} \sum_{n \geq 0} \left(\frac{\coth(\pi(2n+1)\tau)}{\pi(2n+1)^3} + \frac{\tau}{(2n+1)^2 \sinh^2(\pi(2n+1)\tau)} \right) \right\} \quad (16)$$

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} = \pm \frac{\sqrt{\tau}}{2^{3/2}\sqrt{v}} \cdot \exp \left\{ \frac{\pi v}{96\tau^2} + \frac{v\tau}{4\pi} \sum_{m \geq 1} \left(\frac{\tanh(\pi m/2\tau)}{\pi m^3} - \frac{1}{2m^2\tau \cosh^2(\pi m/2\tau)} \right) \right\}, \quad (17)$$

These two expressions enjoy excellent convergence properties in the regimes $\tau \gg 1$ and $\tau \ll 1$, respectively.

C. Asymptotic Expansions for $\tau \gg 1$ and $\tau \ll 1$

From (16, 17) the condensate is seen to behave asymptotically like

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \sim \pm \frac{1}{4\lambda} \exp \left\{ \frac{\lambda^2}{\pi^2} \sum_{n \geq 0} \frac{1}{(2n+1)^3} \right\} = \pm \frac{1}{4\lambda} \exp \left\{ \frac{7\zeta(3)\lambda^2}{8\pi^2} \right\} \quad \left(\begin{array}{c} \text{low temp.} \\ \text{fixed } L \end{array} \right) \quad (18)$$

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \sim \pm \frac{\tau}{2\sigma} \exp \left\{ \frac{\sigma^2}{4\pi^2\tau^2} \sum_{n \geq 0} \frac{1}{(2n+1)^3} \right\} = \pm \frac{\tau}{2\sigma} \exp \left\{ \frac{7\zeta(3)\sigma^2}{32\pi^2\tau^2} \right\} \quad \left(\begin{array}{c} L \rightarrow 0 \\ \text{fixed } T \end{array} \right) \quad (19)$$

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \sim \pm \frac{\tau}{2\sigma} \exp \left\{ \frac{\pi\sigma^2}{192\tau^3} + \frac{\sigma^2}{8\pi^2} \sum_{m \geq 1} \frac{1}{m^3} \right\} = \pm \frac{e^{\zeta(3)\sigma^2/8\pi^2}\tau}{2\sigma} \exp \left\{ \frac{\pi\sigma^2}{192\tau^3} \right\} \quad \left(\begin{array}{c} L \rightarrow \infty \\ \text{fixed } T \end{array} \right) \quad (20)$$

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \sim \pm \frac{1}{4\lambda} \exp \left\{ \frac{\pi\lambda^2}{48\tau} + \frac{\lambda^2\tau^2}{2\pi^2} \sum_{m \geq 1} \frac{1}{m^3} \right\} \sim \pm \frac{1}{4\lambda} \exp \left\{ \frac{\pi\lambda^2}{48\tau} \right\} \quad \left(\begin{array}{c} \text{high temp.} \\ \text{fixed } L \end{array} \right) \quad (21)$$

where only the leading term is kept.

D. Divergence of the quenched chiral condensate in the chiral limit

As noted in the introduction, we are particularly interested in the behavior of the condensate in the chiral limit. The condensate is properly defined by first taking the infinite volume limit and then sending the chirality breaking source to zero. In our framework these two limits are taken simultaneously by sending $\lambda \rightarrow \infty$. This means that in general we do not know *a priori* whether the source is “strong enough” to lead to symmetry breaking when $\lambda \rightarrow \infty$. For example, in QCD(4) if $m \rightarrow 0$ and $L \rightarrow \infty$ such that $mV \rightarrow 0$ in the limit, with V the Euclidean volume, then chiral symmetry breaking will not occur. But if we find a non-zero limiting value for the condensate, we conclude *a posteriori* that the symmetry breaking source is strong enough. Indeed, we shall see that in the (massless) quenched Schwinger model the condensate diverges as $\lambda \rightarrow \infty$ for any temperature.

1. To study the limit of infinite box-length at zero temperature we first perform the limit of zero temperature in (18) which gives the exact result

$$\lim_{\sigma \rightarrow \infty} \frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \Big|_{\lambda \text{ fixed}} = \pm \frac{1}{4\lambda} \cdot \exp \left\{ \frac{7\zeta(3)}{8\pi^2} \lambda^2 \right\}. \quad (22)$$

Clearly the infinite volume limit cannot be taken.

2. To study the limit of infinite box-length at fixed (non-zero, non-infinite) temperature we use (20) from which we see that the condensate now diverges like

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \sim \pm \frac{e^{\zeta(3)\sigma^2/8\pi^2}}{4\lambda} \exp \left\{ \frac{\pi}{24\sigma} \lambda^3 \right\} \quad \left(\begin{array}{l} \lambda \gg 1 \\ \sigma \text{ fixed} \end{array} \right) \quad (23)$$

which is even more virulent than in the zero-temperature case (22).

3. To study the limit of infinite box-length at infinite temperature we first focus on the high-temperature limit at finite box-length. From (21) we find the result

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \sim \pm \frac{1}{4\lambda} \exp \left\{ \frac{\pi \lambda^3}{24\sigma} \right\} \quad \left(\begin{array}{l} \sigma \ll 1 \\ \lambda \text{ fixed} \end{array} \right) \quad (24)$$

which diverges at infinite temperature even at finite box-length.

4. Finally one may check whether the results are significantly altered if the limit of infinite box-length is taken while simultaneously lowering or raising the temperature in such a way that either the aspect ratio $\tau = \sigma/2\lambda$ or the box-volume $v = \sigma\lambda$ stays constant. Under $v \rightarrow \infty$ at fixed τ the condensate is seen from (16) to diverge like

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \sim \pm \text{const} \frac{1}{\sqrt{v}} \exp \left\{ \text{const} v \right\} \quad (25)$$

Under $\tau \rightarrow 0$ at fixed v the condensate is seen from (17) to diverge like

$$\frac{\langle \psi^\dagger P_\pm \psi \rangle}{\mu} \sim \pm \text{const} \sqrt{\tau} \exp \left\{ \text{const} \frac{1}{\tau^2} \right\} \quad (26)$$

Our findings are illustrated in figure 1.

FIGURES

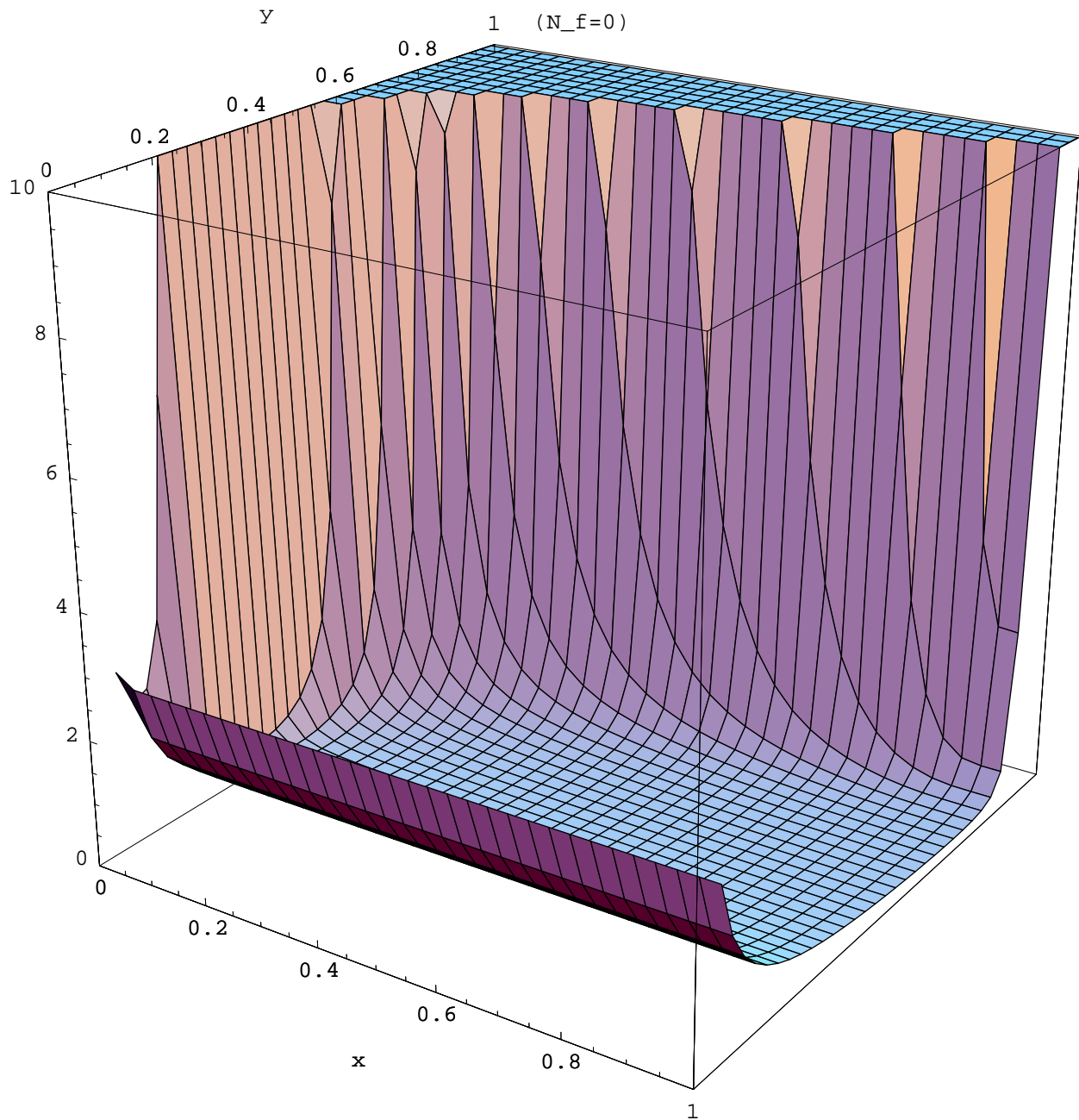


FIG. 1. The quenched condensate at the midpoint as a function of spatial box size and inverse temperature, using eqs. (25, 26). The axes have been rescaled so that the entire range of values is shown: $x = 2/\pi \cdot \arctan(\sigma)$ and $y = 2/\pi \cdot \arctan(\lambda)$, where $\sigma = \mu\beta$ and $\lambda = \mu L$. Unlike the other divergencies, the one near $y=0$ is unphysical as it is an artifact of the boundary conditions. The border of the surface at $x=1$ represents the (massless) quenched Schwinger model at zero temperature with our boundary conditions.

IV. ORIGIN OF INFRARED DIVERGENCES

It is interesting to elucidate the origin of the IR divergences in the quenched theory. Mathematically, things can be traced back to the fact that in two dimensions the gauge-field can be decomposed into a coexact and an exact piece ($A_\mu = -\epsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\chi$ plus, with certain boundary conditions, a topological and a harmonic contribution) and the functional determinant (if present) can be computed by integrating the anomaly. As a consequence, the full effective action in the N_f -flavor Schwinger model is found to read (up to contributions from the topological and harmonic parts)

$$S_{N_f}[\phi] = \frac{1}{2e^2} \int \phi(\Delta^2 - N_f \frac{e^2}{\pi} \Delta) \phi \, dx. \quad (27)$$

Going back to (8, 9, 10) and using decomposition properties of the massless Dirac operator in a given gauge-field (which are specific to two dimensions), the condensate with our boundary conditions and for arbitrary $N_f \geq 0$ is found to factorize

$$\begin{aligned} \langle \psi^\dagger(x) P_\pm \psi(x) \rangle &= \frac{\int dc D\phi \, S_\theta(x, x)_{\pm\pm} e^{-\Gamma_{\theta, N_f}[c, \phi]}}{\int dc D\phi \, e^{-\Gamma_{\theta, N_f}[c, \phi]}} \\ &= \frac{\int dc \, \tilde{S}_\theta(x, x)_{\pm\pm} e^{-N_f \Gamma(c)}}{\int dc \, e^{-N_f \Gamma(c)}} \cdot \frac{\int D\phi \, e^{\mp 2\phi(x) - \Gamma_{\theta, N_f}[\phi]}}{\int D\phi \, e^{-\Gamma_{\theta, N_f}[\phi]}}. \end{aligned} \quad (28)$$

Here $\tilde{S}_\theta(x, x)_{\pm\pm}$ denotes the diagonal entries of the Green function of $i\cancel{D}$ (obeying XB-boundary-conditions) and $\Gamma_{\theta, N_f}[\phi]$ is just $S_{N_f}[\phi]$, up to a term which vanishes for $\theta=0$ (for further details the reader is referred to [14,15]). The first integral in (28) (which is over a c -number) is trivial in the quenched theory. The second integral (which is an integration over all ϕ -fields which are periodic over β and satisfy Dirichlet boundary conditions at $x^1=0$ and $x^1=L$) can be done exactly for all N_f :

$$\frac{\int D\phi \, e^{\mp 2\phi(x) - S_{N_f}[\phi]}}{\int D\phi \, e^{-S_{N_f}[\phi]}} = \exp \left\{ 2e^2 \langle x | \frac{1}{\Delta^2 - N_f \frac{e^2}{\pi} \Delta} | x \rangle \right\}. \quad (29)$$

We stress that the factorization property (28) reflects itself in the form of our analytical results — both in the dynamical case [eqns. (11, 12)] and the quenched case [eqns. (14, 15)]: All expressions consist of a prefactor [the first two lines in eqns. (11, 12) or the factor $1/(4\lambda \sin(\pi\xi))$ in eqns. (14, 15)] and a product of exponentials. In both cases the prefactor corresponds to the first factor in (28), whereas the exponentials represent the second factor in (28), as expressed in (29).

The important point here is that in the quenched case the IR divergences are due entirely to the second factor in (28), i.e. due to the divergence of (29). Indeed, as our explicit expressions show, the first factor in (28) vanishes as a power-law in the limit of infinite box-length in both quenched and full theories. This is as expected, since the leading effect of the chirally asymmetric boundary conditions on the free fermion propagator, $\tilde{S}(x, x)$, is proportional to the propagator from x to the boundary and back, and this propagator falls off since the Dirac operator has a mass-dimension lower than the dimension of the manifold.

By contrast, the operator $\Delta^2 - N_f \mu^2 \Delta$, which appears in the expression (29), has a mass-dimension higher than the dimension of the manifold. Its propagator

$$\mathcal{G}(x - y; N_f) = \langle x | \frac{1}{\Delta^2 - N_f \mu^2 \Delta} | y \rangle, \quad (30)$$

thus grows with distance, and one has the possibility that the boundary conditions become more influential as the spatial length is increased. To understand this in more detail, it is useful to have explicit expressions for the requisite Green functions. In the quenched theory one finds a quadratic increase

$$\mathcal{G}(x - y; N_f = 0) = \frac{1}{8\pi} (x - y)^2 \log(\mu |x - y|) - \frac{1}{8\pi} (x - y)^2, \quad (31)$$

while for $N_f \geq 1$ the result increases only logarithmically

$$\mathcal{G}(x - y; N_f \geq 1) = \frac{1}{N_f \mu^2} \left(-\frac{1}{2\pi} \log(N_f^{1/2} \mu |x - y|) - \frac{1}{2\pi} K_0(N_f^{1/2} \mu |x - y|) \right). \quad (32)$$

The less rapid increase in the unquenched case turns out, as shown by the explicit results, to be insufficient to lead to an IR divergence in the condensate.

One can push the argument a little further by imagining using images to enforce the boundary conditions. One cannot actually repeat our calculation in this way, because the sum over the two-fold infinite set of images does not converge, since contributions grow as the images become more distant — both in quenched and full theories. But one can use images to obtain the pattern of IR divergences observed in our results by considering the simpler geometry of a semi-infinite spatial extent at zero temperature: at $x^1 = 0$ we apply XB-boundary conditions whereas for $x^1 \rightarrow \infty$ we require L^2 -integrability. The analogous quantity to that which we have calculated is the condensate at a distance $L/2$ from the left boundary, and the important issue is how it behaves as $L \rightarrow \infty$. In this new geometry we need only a single image, and it is the Green function between the original position and its image which dominates for large L . The asymptotic forms are thus

$$\exp \left\{ 2e^2 \langle x | \frac{1}{\Delta^2 - N_f \frac{e^2}{\pi} \Delta} | x \rangle \right\} = \begin{cases} \exp \left\{ \text{const} \cdot \lambda^2 \log(\lambda) \right\} & (N_f = 0) \\ \exp \left\{ \text{const} \cdot \log(\lambda) \right\} & (N_f \geq 1) \end{cases} \quad (33)$$

where we use $\lambda = L|e|/\sqrt{\pi}$ instead of L to aid the comparison with our previous results. We expect that our results should have a similar form, but with different constants, due to the other mirror copies present in our geometry.

This expectation turns out to be correct, up to logarithmic accuracy. For the quenched case, $N_f = 0$, our exact result at zero temperature, eqn. (22), takes the form $\exp(0.107\lambda^2)$. It thus matches eqn. (33), except that the logarithm is absent. For $N_f = 1$ the form (33) works with “const” equal to 1, which is exactly what is needed to cancel the asymptotic $1/L$ -behavior of the first factor in (28) and to reproduce the well-known value for the condensate of the one-flavor Schwinger model in the limit $L \rightarrow \infty$ [14]. For $N_f = 2$ the “const” is found to be $1/2$, which is not sufficient to cancel the asymptotic $1/L$ -behavior of the first factor in (28). Consequently the two-flavor condensate vanishes as $1/\sqrt{L}$ when the model is quantized with XB-boundary conditions [15].

This completes our attempt to make sense out of the functional form of our analytical results. While it is interesting to see that the IR-sickness of the quenched Schwinger model can be traced back to the *long-distance behavior* of the Green function $\langle x|\Delta^{-2}|y\rangle$, it is clear that the argument is specific to two dimensions.

V. SPECULATIONS ON THE MASSIVE THEORY

Having an analytical result for the (massless) quenched Schwinger model it is interesting to consider the IR behavior of the corresponding massive model, quenched QED(2). In particular, we would like to know what happens if one disentangles the limits $L \rightarrow \infty, m \rightarrow 0$ and takes them in the standard order: first $L \rightarrow \infty$ and then $m \rightarrow 0$. In QCD(4) these two limits do not commute: the standard order leads to a non-zero condensate, whereas $\langle \bar{\psi}\psi \rangle$ vanishes when $m \rightarrow 0$ is taken at any finite L . In quenched QED(2) knowledge concerning these two situations is asymmetric. Taking the massless limit first, analytic results are obtained and, as we have seen, one cannot take the $L \rightarrow \infty$ limit when $m=0$. The result of taking the limits in the standard order is, however, controversial. There is no exact analytical expression for the condensate for $m \neq 0$, and so one must resort to approximate methods. Reference [11] argues that sending $L \rightarrow \infty$ at fixed m yields a finite result for the condensate, and that this result remains finite as $m \rightarrow 0$. This is what is found explicitly in the approximate calculation of Ref. [18]. On the other hand Ref. [19] proposes a power-law divergence as $L \rightarrow \infty$. In the following we analyze the situation using two approaches. First, we expand in powers of the quark mass, and find that each coefficient diverges exponentially as the volume is sent to infinity. Second, we generalize the approach of Smilga [12], and argue that there is in fact no IR divergence at finite m , and that the IR divergences in our first approach are artifacts of expanding about $m=0$. The IR divergence does, however, reappear as a power law divergence when $m \rightarrow 0$, in disagreement with Refs. [11,18].

The derivatives of the condensate with respect to m at $m=0$ are higher correlation functions in the massless theory, and can be computed analytically. Note that, since our boundary conditions break chiral symmetry, there is no reason to expect non-analyticity about $m=0$ for finite volume.

We start from the form of the condensate in the massive quenched theory

$$\langle \psi^\dagger P_+ \psi \rangle(x) = \frac{\int dc D\phi (S_{\theta,m}(x,x)_{++} - \tilde{S}_{\theta,m}(x,x)_{++} + \tilde{S}_{\theta,0}(x,x)_{++}) e^{-\frac{1}{2e^2} \int \phi \Delta^2 \phi}}{\int dc D\phi e^{-\frac{1}{2e^2} \int \phi \Delta^2 \phi}}, \quad (34)$$

where $S_{\theta,m}$ is the inverse of $\not{D}+m$ with XB boundary conditions, and $\tilde{S}_{\theta,m}$ its counterpart at zero gauge-coupling. The subtractions are chosen such that (34) vanishes in the limit $e \rightarrow 0$ at fixed m , while still reducing to (28) when $m \rightarrow 0$ at fixed e . In addition, they make the expression UV-finite. To get a sense of the general term in the expansion about $m=0$, we consider the first derivative,

$$\begin{aligned} \left. \frac{d\langle \psi^\dagger P_+ \psi \rangle(x)}{dm} \right|_{m=0} &= \frac{1}{2} \sum_{\pm} \int dy \int dc \tilde{S}_{\theta}(x,y)_{+\pm} \tilde{S}_{\theta}(y,x)_{\pm+} \times \\ &\quad \left(\frac{\int D\phi e^{-2\phi(x) \mp 2\phi(y) - \frac{1}{2e^2} \int \phi \Delta^2 \phi}}{\int D\phi e^{-\frac{1}{2e^2} \int \phi \Delta^2 \phi}} - 1 \right) \end{aligned}$$

$$= \frac{1}{2} \sum_{\pm} \int dy \int dc \tilde{S}_{\theta}(x, y)_{\pm\pm} \tilde{S}_{\theta}(y, x)_{\pm\pm} \times \quad (35)$$

$$\left(\exp \left\{ 2e^2 \left(\langle x | \frac{1}{\Delta^2} | x \rangle \pm \langle x | \frac{1}{\Delta^2} | y \rangle \pm \langle y | \frac{1}{\Delta^2} | x \rangle + \langle y | \frac{1}{\Delta^2} | y \rangle \right) \right\} - 1 \right) .$$

Here y runs over the entire manifold, the c -integration is the average over a c -number valued harmonic part of the gauge field (necessary since the cylinder is not simply connected [14]), and the sum consists of two terms (i.e. the \pm are either all $+$ or all $-$). The explicit forms of \tilde{S}_{θ} (the free massless propagator subject to the XB-boundary-conditions) and $\langle x | \Delta^{-2} | y \rangle$ are given in [14]. What is important here is that, for geometrically fixed x and y , $\langle x | \Delta^{-2} | y \rangle$ diverges as L^2 . Thus, for at least one of the choices of sign, the second factor in (35) diverges as e^{L^2} . This dominates the power-law fall-off of the first factor. Thus there is at least a set of y -values with nonzero measure for which the combined argument of the y -integration in (35) is IR-divergent³. Similar arguments can be made for the higher derivatives of the condensate.

We conclude that each coefficient in the expansion of the condensate about $m = 0$ is highly IR divergent. Of course, this does not necessarily imply that the quenched condensate diverges when $V \rightarrow \infty$ at fixed m , because the limits may not commute. For example if the quenched condensate had the following schematic dependence on m and V

$$\langle \psi^\dagger P_{\pm} \psi \rangle \simeq \frac{1}{m + \exp(-V)} \quad , \quad (36)$$

then it would be finite as $V \rightarrow \infty$ at fixed m , while derivatives at $m = 0$ are all IR divergent. In the following we argue that a form like eq. (36) actually holds.

We use the line of reasoning initiated by Smilga [12], generalized to the quenched case. We briefly recapitulate his argument. The key result (valid for $N_f \geq 1$ and for $N_f = 0$) is

$$\langle \sum_n \frac{1}{\lambda_n^2} \rangle = \frac{1}{2\pi^2} \int d^2x d^2y \frac{1}{(x-y)^2} \exp\{2e^2(\mathcal{G}(0; N_f) - \mathcal{G}(x-y; N_f))\} . \quad (37)$$

This relates the sum of the inverse-squares of the eigenvalues⁴ of the Dirac operator to the Green function appearing in (31, 32). As $V \rightarrow \infty$ the leading behavior of the r.h.s. of (37) is determined by the asymptotic form of \mathcal{G} . For unquenched theories one finds, using (32),

$$\langle \sum_n \frac{1}{\lambda_n^2} \rangle \propto V^{(N_f+1)/N_f} \quad (N_f \geq 1) \quad , \quad (38)$$

with missing dimensions provided by powers of e . This implies that the characteristic size of the lowest eigenvalues is

³On this point the unquenched theory differs significantly: Each Green function diverges only logarithmically with L , and the overall divergence of the second factor is compensated ($N_f = 1$) or overwhelmed ($N_f \geq 2$) by the fall-off of the c -integral.

⁴In the remainder of this section we use λ to denote eigenvalues and not the dimensionless box length.

$$\lambda_c \propto V^{-(N_f+1)/(2N_f)} \quad (N_f \geq 1) \quad . \quad (39)$$

One now assumes that, in the limit of infinite V , one can define a density of eigenvalues per unit volume, $\rho(\lambda)$. If so, then one has approximately that

$$\rho(\lambda_c) \approx \frac{1}{\lambda_c V} \propto \lambda_c^{(N_f-1)/(N_f+1)} \quad (N_f \geq 1) \quad . \quad (40)$$

Since this holds for all large V , one can read off the small λ dependence of $\rho(\lambda)$. One can check this result by noting that it reproduces the initial result (38):

$$\langle \sum_n \frac{1}{\lambda_n^2} \rangle \approx \int_{\lambda_c}^{\lambda_{\max}} V \rho(\lambda) \frac{1}{\lambda^2} \sim V \lambda_c^{-2/(N_f+1)} \propto V^{(N_f+1)/N_f} \quad . \quad (41)$$

Here we approximate finite volume by the lower cut-off on the λ integral.

This line of reasoning can be extended to include the quenched case, $N_f = 0$. Smilga notes that if one uses the quenched result for $\mathcal{G}(x)$, eq. (31), one finds that, asymptotically, the sum rule is

$$\langle \sum_n \frac{1}{\lambda_n^2} \rangle \propto \exp\{V\} \quad (N_f = 0) \quad , \quad (42)$$

and so the characteristic size of the lowest eigenvalues is exponentially small

$$\lambda_c \propto \exp\{-V\} \quad (N_f = 0) \quad . \quad (43)$$

We stress that in both these results we control neither the constant multiplying V in the exponent (aside from the fact that it is proportional to e^2), nor subleading power dependence multiplying the exponentials.

At this point we extend the results of Smilga by determining the form of $\rho(\lambda)$ in the quenched theory ⁵. The same argument used above leads to

$$\rho(\lambda_c) \approx \frac{1}{\lambda_c V} \sim \frac{-1}{\lambda_c \log(\lambda_c)} \quad . \quad (44)$$

This cannot be completely correct because it predicts that the number of eigenvalues in an interval,

$$n(\lambda_2) - n(\lambda_1) = \int_{\lambda_1}^{\lambda_2} d\lambda \rho(\lambda) = \log[\log \lambda_2 / \log \lambda_1] \quad , \quad (45)$$

diverges as $\lambda_1 \rightarrow 0$. We propose instead the convergent form (with missing dimensions supplied by factors of e)

$$n(\lambda) \propto -\frac{1}{\log \lambda} \quad \Rightarrow \quad \rho(\lambda) \propto \frac{1}{\lambda (\log \lambda)^2} \quad . \quad (46)$$

⁵We acknowledge a useful discussion with A. Smilga on this point.

This satisfies $n(\lambda_c) \propto 1/V$, which is an alternative criterion for extracting the infinite volume eigenvalue density from the finite volume results ⁶. In fact, the precise form of $\rho(\lambda)$ is not important: all that we need is that it falls off as $1/\lambda$, up to logarithmic corrections, and that these corrections make it integrable as $\lambda \rightarrow 0$.

We now insert this result into the expression for the condensate, approximating the effect of finite volume by a lower cut-off:

$$\langle \psi^\dagger P_\pm \psi \rangle \approx \int_{\lambda_c}^{\infty} d\lambda \frac{2m}{\lambda^2 + m^2} \rho(\lambda) . \quad (47)$$

There are two cases to consider. If $m < \lambda_c \sim e^{-V}$, then the IR cut-off is provided by λ_c

$$\langle \psi^\dagger P_\pm \psi \rangle \approx \int_{\lambda_c}^{\infty} \frac{2m}{\lambda^3} \propto \frac{m}{\lambda_c^2} \quad (m < \lambda_c) . \quad (48)$$

If $m > \lambda_c$ then the mass provides the IR cut-off, and one has

$$\begin{aligned} \langle \psi^\dagger P_\pm \psi \rangle &\approx \int_{\exp(-V)}^m d\lambda \frac{2}{m} \rho(\lambda) + \int_m^{\infty} \frac{2m}{\lambda^3} \\ &= \frac{2[n(m) - n(\lambda_c)]}{m} + \frac{1}{m} \\ &\sim \frac{1}{m} [1 + O(1/V)] \quad (m > \lambda_c) , \end{aligned} \quad (49)$$

where we have dropped subleading logarithms, which are not controlled. In other words, the condensate starts out at zero when $m = 0$, as required in finite volume with no chiral symmetry breaking, rises linearly until $m \simeq \lambda_c \sim e^{-V}$, and then falls as $1/m$ up to logarithms. The point we wish to stress is that eqs. (48, 49) have no IR-divergences. Sending the volume to infinity at finite m , one ends up with a finite value. If one then sends $m \rightarrow 0$, however, the IR-divergence returns as a $1/m$ divergence.

The preceding argument applies for boundary conditions which do not break chiral symmetry, and thus is not directly applicable to the massive theory with chirality breaking boundary conditions. Nevertheless, since a non-zero mass removes the IR divergences with chirally symmetric BC, it is plausible that it will also do so with chirality breaking BC. If so, the IR divergences we found in the mass-derivatives of the condensate must be special to $m = 0$. We conclude this section by presenting a simple model which leads to this result.

We propose that, when considering the condensate far from the chirality breaking boundaries, their effects can approximately be represented by using Smilga's analysis but with a small, volume dependent, shift in the mass. More precisely, we suggest that one can use eqs. (48, 49) with

$$m \rightarrow m_{XB} = m + \lambda_c \sim m + e^{-V} . \quad (50)$$

The shift by λ_c is chosen so as to reproduce the form of our exact results. In particular, it puts us in the region where (49) holds, so that

⁶In the unquenched example above this criterion gives the same result for $\rho(\lambda)$ as obtained in eq. (40).

$$\langle \psi^\dagger P_\pm \psi \rangle_{XB} \simeq \frac{1}{m_{XB}} = \frac{1}{m + \exp(-V)} . \quad (51)$$

This simple form agrees with all our results and expectations: when $m = 0$ we recover the exponential IR divergence of our exact result; when $V \rightarrow \infty$ at fixed m there is no IR divergence; and each successive derivative, evaluated at $m = 0$, is more IR divergent. This shows explicitly how the perturbative argument presented above can be correct but completely misleading.

VI. SUMMARY

We have presented a study of the chiral condensate in the (massless) quenched Schwinger model at finite temperature and with bag-inspired spatial boundary conditions which play the role of a small fermion mass. Our main analytical result is given in eqs. (14, 15).

Our first finding is that the (massless) quenched Schwinger model is ill-defined due to an infrared embarrassment: At finite box-length with the XB- boundary conditions applied at the two spatial ends, it is a well-defined theory, but the condensate shows a singular behavior when the boundaries are sent to infinity. The precise form of this divergence depends on the details how the volume of the manifold is sent to infinity, but the generic structure is “power-law fall-off times exponential divergence”.

Our second observation is that the condensate in the (massless) quenched Schwinger model does not “melt” at high temperatures, instead diverging in the limit of infinite temperature. This has to be contrasted to the case $N_f \geq 1$ where the Schwinger model is known to show the regular behavior, i.e. a condensate which decreases when the temperature gets large (see e.g. [17,15] and references therein).

Both predictions can be checked using numerical simulations on a sufficiently large lattice, and an interesting step in this direction has recently been taken in Ref. [20].

From a conceptual point of view it should be emphasized that our results for the massless case represent exact analytic findings. They stem from a textbook-style analytical evaluation of the path integral with fields subject to the constraints imposed by the boundary conditions. The latter break chiral symmetry explicitly and prevent exact zero modes of the massless Dirac operator. This shows that the IR divergence of the condensate in the quenched massless theory is not tied to a zero in the spectrum of the Dirac operator on a certain class of gauge field configurations.

On the other hand, our results for the massive theory are more speculative. By extending the approach of Smilga [12] we argue that the quenched condensate is IR finite, and reconcile this with our finding that the derivatives of the condensate with respect to the quark mass are IR divergent at $m=0$.

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